Basic Formulas:

To be consistent with the altitude reported by the aircraft altimeters, we should use the International Standard Atmosphere or the equivalent U.S. Standard Atmosphere. In our altitude range, it is define as follows:[1]

- reference level z=0 at p_0=1013.25 mb and T_0 =15°C.
- lapse rate of -6.5°C/km or $\lambda = -0.0065°C/m$ below 11,000 m; lapse rate 0 above 11,000 m

The fundamental equation that gives pressure change with height (h) is

$$\frac{dp}{p} = -\left(\frac{g}{RT}\right)dh \tag{1}$$

where g=9.80665 m/s^2 and R=287.0531 J/(kg K). The values of g, R* = 8.31432x10^3, M$_d$=28.9644, as well as p_0, T_0, and λ as given above, are specified as part of the standard atmosphere so should be taken as exact[2]; R=R*/M$_d$=287.0531 is then accurate to seven significant figures. (This note will give numerical results with seven significant figures, to preserve six-figure accuracy in results.) With a lapse rate of λ defined as dT/dh where h is altitude, the equation becomes

$$\frac{dp}{p} = -\left(\frac{g}{\lambda R}\right)\frac{dT}{T} \tag{2}$$

which integrates to

$$p = p_0\left(\frac{T_0}{T_0+\lambda h}\right)^{\frac{g}{\lambda R}} \tag{3}$$

or, after rearranging to obtain h as a function of p,

$$h = \frac{T_0}{\lambda}\left[\left(\frac{p}{p_0}\right)^{-\lambda R/g} - 1\right] = a_z\left[1-\left(\frac{p}{p_0}\right)^{b_p}\right] \tag{4}$$

where $a_z = -T_0/\lambda$ is 44330.77 m and $b_p = -\lambda R/g$ is 0.1902632 (dimensionless).

Above 11,000 m, the temperature in the standard atmosphere becomes constant (at T_1=288.15-11*6.5=216.65K=-56.5°C.). Then (1) integrates to

[1] see "U.S. Standard Atmosphere, 1976", NASA-TM-A-74335, available for download at http://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19770009539_1977009539.pdf

[2] The quoted value of R* is not exactly equal to the currently accepted values of the more fundamental quantities k and N_A,respectively the Boltzmann constant and Avogadro's Number, the product of which should give R*. The difference is in the 5th significant digit. However, the value as quoted should still be used for pressure altitude calculations because it is adopted as part of the standard.

$$\ln\left(\frac{p}{p_1}\right) = -\frac{g(h-h_1)}{RT_1} \tag{5}$$

where p_1 at h_1=11,000 m can be found from (3): $p_1 = 226.3206$ mb. Then, solving (5) for h gives

$$h = h_1 - \frac{RT_1}{g}\ln\left(\frac{p}{p_1}\right) = h_1 - c_z \ln\left(\frac{p}{p_1}\right) \tag{6}$$

where $c_z = (RT_1/g) = 6341.620$ m.

Comparison to existing code:

The formula in use for the C-130 is:

```
palt = 153.77 * SFCT * (1.0 - pow((double)psxc / ASTG, 0.190284));
```

where SFCT is 288.15 and ASTG is 1013.248. The product 153.77*SFCT gives 44,308.8 m, while I think this should be 44,330.77 as derived above. I suspect the lower number was obtained by using a reference temperature of 288K, rather than the correct value of 288.15K. The exponent, 0.190284, is also slightly different from the value of b_p derived above, 0.1902632. This difference is not significant, and the use of ASTG=1013.248 is also insignificantly different from the correct value of 1013.25 mb, but there is no reason to use the wrong value because the reference pressure altitude includes 1013.25 as the exact reference pressure for the standard atmosphere.

The formula in use for the GV is

```
palt = psxc / ASTG;
    if (palt >= 0.223198605)
        palt = 44308.0 * (1.0 - pow(palt, 0.190284));
    else
        palt = 11000.0 + 14600.0 * log10(0.223198605 / palt);
```

The pressure ratio for the break corresponds to a pressure of 0.2231986*1013.248=226.156 vs the value p_1=226.3206 mb derived above – an insignificant difference. The constant 44308.0 should be changed to 44330.77, and the exponent to 0.1902632, as developed above. For the high-altitude branch, the formula for the additional altitude above 11000 m gives the result in terms of the ratio of pressure to the sea-level reference pressure rather than the transition pressure as used in the above derivation, and uses a base-10 logarithm. In terms of a base-10 logarithm, the value 14600 would become 14600/$\ln_e$(10)=6340.70 m if the logarithm were expressed in natural logarithms. The factor 0.223198605 converts the pressure ratio relative to the sea-level reference pressure to a ratio relative to the transition pressure: the correct value $p_1/p_0 = 0.2233611$ is insignificantly different from the conversion factor used. The value 6340.70 is only slightly different from the value derived above, 6341.620 m. To use the log10() function, the constant should be 14602.12 m [6341.620*ln(10)] according to the derivation in the first section.

Recommendation:

Use only one formula for all aircraft.[3] (The reference atmosphere for all is the same; it is only the pressure range that is different for different aircraft.) The recommended formula is the current HIAPER branch modified as follows (with some changes made just for clarity, to help whoever looks at this next):

```
          // transition pressure at the assumed ISA tropopause:
#define ISAP1 226.3206
          // reference pressure for standard atmosphere:
#define ISAP0 1013.25
if (psxc > ISAP1)
    palt = 44330.77 * (1.0 - pow(psxc/ISAP0, 0.1902632));
else
    palt = 11000.0 + 14602.12 * log10(ISAP1/psxc);
```

The effects of the changes are small but not entirely negligible, and this change will make us consistent with the adopted standards for aviation. The differences are as follows:

1. The change in the constant for the low-altitude branch, to 44330.77 m, causes about a 5 m maximum change in results, with the maximum at 11000 m.
2. The change in the exponents for the low-altitude branch, to 0.1902632, causes at most a change of about 1 m in the results.
3. The change in the high-altitude branch increases with altitude, but at 120 mb (about 15 km) the change is about +5 m.

The recommended change to Bulletin 9 is then as follows:

ISA Pressure Altitude (m): <u>PALT</u>
The derived altitude obtained from the reference barometric (static) pressure measurement using the International Standard Atmosphere (ISA), equivalent to the reference atmosphere for aviation operations worldwide.[4] The pressure altitude is best interpreted as a variable equivalent to the measured pressure, not as a geometric altitude. In the following description of the algorithm, some constants (identified by the symbol †) are specified as part of the ISA and so should not be "improved" to more modern values (e.g., R_0').[5]

[3] This formula will need further modification if it is ever used above 20 km, where the lapse rate again changes in the standard atmosphere; that change is not incorporated here.

[4] see "U.S. Standard Atmosphere, 1976", NASA-TM-A-74335, available for download at http://ntrs.nasa.gov/archive/nasa/casi.ntrs.nasa.gov/19770009539_1977009539.pdf

[5] Prior to and including some projects in 2010, processing used slightly different coefficient (perhaps this should be documented here?)

T_0= 288.15 K, reference temperature†
λ = -0.0065 K/m = the lapse rate for the troposphere†
P_s = measured static pressure, hPa, usually from PSXC
P_0 = 1013.25 hPa, reference pressure for PALT=0†
M_d = 28.9644 $kg/kmol$= molecular weight of dry air, ISA definition†
g = 9.80665 m s^{-2}, acceleration of gravity†
R'_0 = universal gas constant, defined as 8.31432×10^3 $J\,\mathrm{kmol}^{-1}\,\mathrm{K}^{-1}$†
z_T = altitude of the assumed tropopause = 11,000 m†
$x = R'_0\lambda/(M_d g) \approx 0.1902632$ (dimensionless)[a]

For pressure > 226.3206 hPa (equivalent to a pressure altitude < z_T):

$$\mathrm{PALT} = -\left(\frac{T_0}{\lambda}\right)\left(1-\left(\frac{P_s}{P_0}\right)^x\right)$$

otherwise, if T_T and p_T are respectively the temperature and pressure at the altitude z_T:

$$T_T = T_0 + \lambda z_T = 216.65\,\mathrm{K}$$

$$p_T = p_0\left(\frac{T_0}{T_T}\right)^{\frac{gM_d}{\lambda R'_0}} = 226.3206\,\mathrm{hPa}$$

$$\mathrm{PALT} = z_T + \frac{R'_0 T_T}{gM_d}\ln\left(\frac{p_T}{p_s}\right)$$

which is coded as follows (with conversion between natural and base-10 logarithm):

```
// transition pressure at the assumed ISA tropopause:
#define ISAP1 226.3206
          // reference pressure for standard atmosphere:
#define ISAP0 1013.25
if (psxc > ISAP1)

    palt = 44330.77 * (1.0 - pow(psxc/ISAP0, 0.1902632));
else
    palt = 11000.0 + 14602.12 * log10(ISAP1/psxc);
```

[a] This is the rounded value used for data processing.