Subject: Recommendation re a geopotential-height variable
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Background

The geopotential height is used widely in meteorology. It is the height variable plotted on constant-pressure weather maps like the 500-mb map, and radiosonde and dropsonde measurements are reported in terms of geopotential height. The reason is that, because the acceleration of gravity g varies with latitude, there is a component of g causing acceleration equatorward along surfaces of constant height but g is always perpendicular to surfaces of constant geopotential height. Pressure fields plotted on surfaces of constant geopotential height, or equivalently geopotential height fields plotted on surfaces of constant pressure, are not affected by these non-perpendicular components of g and so provide an undistorted depiction of dynamic effects. For similar reasons, the definition of D-value is the difference between geopotential height and pressure altitude. We now calculate this as GGALT−PALT, but GGALT is not geopotential height but rather geometric height above sea level so D-value fields as now defined are distorted if they span any significant range in latitude. We should correct this.

It would therefore be useful to add a variable representing geopotential height. The proposal here is to add two variables, GEOPTH and GGHWGS, the first representing geopotential height and the second representing height above the WGS84 reference ellipse.

Definitions

The geopotential $\Phi(h,\lambda)$ is the gravitational potential energy per unit mass at altitude h and latitude λ, relative to the potential energy at the same latitude at mean sea level (MSL). The geoid, a smoothed global representation of MSL, is a surface of constant geopotential and so can be taken as the zero reference globally. The gravitational field of the earth varies with latitude and decreases with height, so geopotential is a function of these variables. In terms of H, the height above the geoid, the geopotential is calculated as

$$\Phi(H,\lambda) = \int_0^H g(z',\lambda)dz' \ , \tag{1}$$

using a representation of the acceleration of gravity ($g(z',\lambda)$ as defined below.

Some other terms used below are defined here, along with symbols used here to represent them:

orthometric height (H): The height above the geoid. This is the measurement provided by GGALT from the GPS units on NCAR research aircraft. This is sometimes called geometric height or height above mean sea level (MSL). The geoid is a smoothed representation of mean sea level, but there may be local variations from effects like ocean circulations or small-scale gravitational anomalies. Nevertheless, it is the normal reference for measurements labeled “MSL”.

height above ellipsoid (h): Height above the WGS84 reference ellipsoid. This is the height normally measured directly by GPS receivers, sometimes called h_{WGS}. The reference ellipsoid is an ellipsoid of revolution that provides a generalized representation of an equipotential surface, but it differs from the more variable geoid that provides a better representation of an equipotential surface.

geoid height (Δ): The height from the reference ellipsoid to the geoid. The GPS units on the research aircraft report this as GGEOIDHT. $h = H + \Delta$.

geopotential_height (Z): The height where the geopotential of a parcel in a constant gravitational field of $g_0 = 9.80665\,\mathrm{m\,s^{-2}}$ would be the same as the actual geopotential of the parcel.

pressure_altitude: The geopotential height having a given pressure in a reference atmosphere defined as the International Standard Atmosphere (ISA). The ISA is defined in terms of a specified temperature profile vs. geopotential height for a dry atmosphere. The pressure altitude is not a real altitude and should be regarded as a convenient re-labeling of pressure.

d-value: The difference between the geopotential height and the pressure altitude. This variable is particularly useful when mapping pressure fields from an aircraft because it removes the first-order effects of small fluctuations in flight level during mapping.

The Basic Equation for Geopotential Height

The equation used in Ranadu::GeoPotHeight()

The dependence of the gravitational acceleration on latitude and altitude is represented well by the ellipsoidal gravity formula of the WGS84, using the Somigliani equation (see this URL) modified to include altitude dependence:

$$g(h,\lambda) = g_e\left(\frac{1+g_1\sin^2(\lambda)}{(1-g_2\sin^2\lambda)^{1/2}}\right)(1-(k_1-k_2\sin^2(\lambda))h+k_3h^2) \tag{2}$$

where $g_e = 9.780327\,\mathrm{m\,s^{-2}}$ is the reference value at the ellipsoid and at the equator, $g_1 = 0.001931851$ and $g_2 = 0.006694380$. (An approximation used previously was to represent the decrease in gravity with altitude by the coefficient $a_z = -3.086\times10^{-6}\,\mathrm{m^{-1}}$, which is an approximation to this better representation.) Gravity as represented in this way includes the effects of the gravitational field and also the rotation of the earth, with a latitude dependence at the reference ellipsoid given by the first factor in parentheses on the right side of this equation. The coefficients $\{k_1, k_2, k_3\}$ are {3.1570428706e-07, 2.1026896504e-09, 7.3745167729e-14} in SI units. Comments regarding the source of these coefficients are included in the Ranadu::Gravity() source file. See the above-referenced URL for the origins of those coefficients, which in the case of altitude dependence represent leading terms in a Taylor expansion.

There is a sublety in use of this equation that is usually ignored: The reference level is the WGS84 ellipsoid, not the geoid. The problem arises from the usual definitions of geopotential or geopotential altitude (below), which reference the potential energy relative to MSL (i.e., the geoid). To find the geopotential at a given orthometric (geometric) altitude H, one should therefore integrate (2) from $h = \Delta$ to $h = H + \Delta$, where Δ may be either a positive or negative quantity. Because the altitude dependence is weak and the magnitude of Δ is typically smaller than 100 m, this makes little difference in the result, as will be demonstrated below.

The Taylor-series representation of the altitude dependence in (2) is integrable analytically, so given an orthometric altitude H (e.g., from GGALT), the result for the geopotential is

$$\Phi(H,\lambda) = g_e \left(\frac{1 + g_1 \sin^2 \lambda}{(1 - g_2 \sin^2 \lambda)^{1/2}} \right) \left(H - \frac{((H+\Delta)^2 - \Delta^2)}{2} (k_1 - k_2 \sin^2 \lambda) + \frac{((H+\Delta)^3 - \Delta^3)}{3} k_3 \right) \tag{3}$$

The leading term is unaffected by the adjustment represented by Δ, but the second and third terms are changed (if Δ is small compared to H) by approximately $H\Delta(-k_1 + k_2 \sin^2(\lambda)) + k_3 H)$ which, for $H = 15{,}000\,\mathrm{m}$ and $\Delta = 100\,\mathrm{m}$, represents a change in the right-side factor representing the height integration of magnitude smaller than about 0.5 m or about 0.003%. The result is a similar change in geopotential height as derived below, so the adjustment of orthometric height by Δ can be ignored with almost negligible change in the answer.

From the definition of geopotential altitude,[1]

$$\begin{aligned} Z(H,\lambda) = \frac{\Phi(H,\lambda)}{g_0} = \frac{1}{g_0} \Bigg\{ & g_e \left(\frac{1 + g_1 \sin^2 \lambda}{(1 - g_2 \sin^2 \lambda)^{1/2}} \right) \\ & \times \left(H - \frac{1}{2} \left((H+\Delta)^2 - \Delta^2 \right) (k_1 - k_2 \sin^2 \lambda) + \frac{1}{3} \left((H+\Delta)^3 - \Delta^3 \right) k_3 \right) \Bigg\} \end{aligned} \tag{4}$$

where H is the orthometric or geometric altitude. Usually, Δ can be set zero with little effect. To include it requires knowledge of the height of the geoid above the reference ellipsoid. This is a complicated relationship that can be looked up from various data sources and which, at low resolution, is included in the data stored in the GPS receiver and provided for recording. Therefore the most straightforward way to make this conversion is to use the variable recorded as GGEOIDHT in (4). Another alternative is to use a calculator like that at this URL to find the height of the geoid.

[1] It is useful, although an aside, to consider how geopotential altitude is calculated from a radiosonde sounding. The difference in geopotential height between two pressure surfaces is:

$$\Delta H_{12} = -\frac{R_d}{g_0} \int_{p_1}^{p_2} T_V(p) d \ln p$$

which does not require knowledge of the local gravity because that local gravity is represented by the pressure gradient through the hydrostatic equation. The integration can start from the known geopotential height of the launch site and the measured station pressure.

A few checks indicate agreement between these two data sets to typically within about 0.5 m. At this web site, you can also provide an ASCII file containing latitude and longitude for many points and it will calculate the height of the geoid for all of these, so that is an alternative to processing in cases where the GGEOIDHT variable is not present in the archived files.

Alternate approach – MJ Mahoney

The following is an alternate approach based on the method developed by MJ Mahoney for finding the geopotential altitude given the geometric altitude. It was documented at a JPL web site that is no longer maintained, but EOL preserved a copy at this URL. The general approach is also documented below and is supported via a function embedded in this document:

1. Differentiate (2) with respect to Z and evaluate at Z=0:

$$\frac{dg(Z,\lambda)}{dZ} = -g_e F(\lambda)\alpha(\lambda) \tag{5}$$

   where $F(\lambda)$ is the first factor in parentheses in (2) and $\alpha(\lambda) = k_1 - k_2 \sin^2(\lambda)$.

2. Use the known dependence of g on geometric distance, $g = g' r^2/(r+z)^2$. This leads to the geopotential, the integrated value from 0 to z, of

$$\Phi(z) = \int_0^Z g(z)dz = -g' \frac{rz}{r+z} \tag{6}$$

3. Regard (5) as defining an appropriate radius that will characterize the decrease in g with z:

$$\frac{dg(z,\lambda)}{dz} = -g' \frac{2}{(r+z)} = -g_e F(\lambda)\alpha(\lambda)$$

   which, evaluated at Z=0, gives $r = 2/\alpha(\lambda)$. This suggests using r as a radius representing the decrease in gravity with height, such that the geopotential is given by

$$\Phi(\lambda,z) = g_e F(\lambda) \frac{zr}{(r+z)} = g_0 Z$$

   where r is not a real radius of the Earth but instead a parameter that represents the known height dependence of gravity. This leads to this equation for the geopotential height:

$$Z = \frac{g_e}{g_0} F(\lambda) \frac{zr}{(r+z)}$$

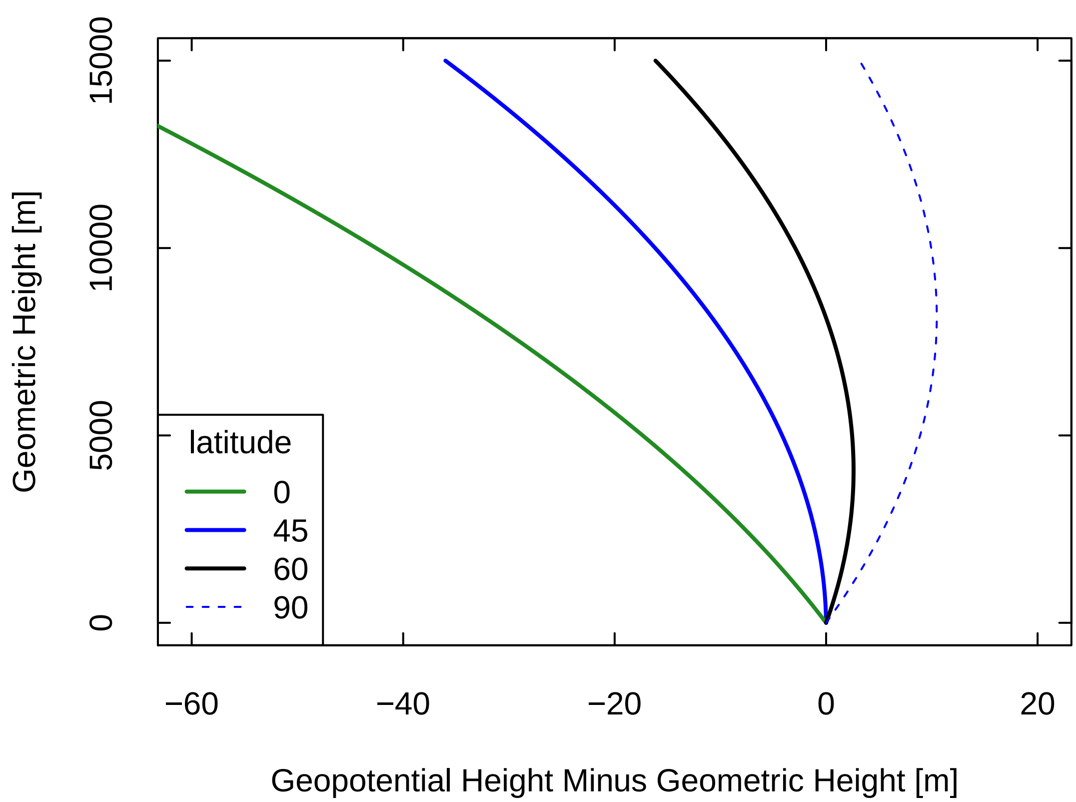
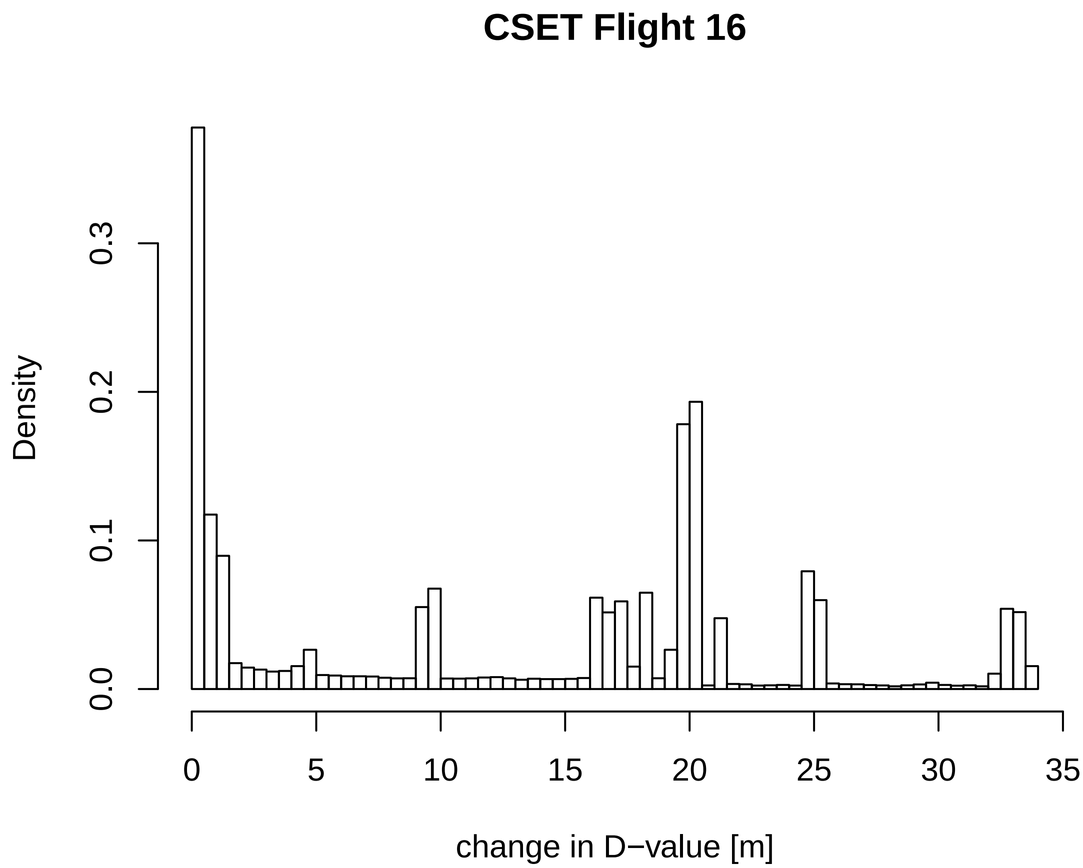


Figure 1: Difference between geopotential and geometric height for several latitudes, if the geoid offset is neglected.

Evaluations indicate near-exact agreement within numerical precision with the function described above, provided the geoid adjustment is set zero in the latter. Note, however, that this alternate approach matches the derivative provided by the first two terms in the Taylor expansion but is insensitive to the third term with coefficient k_3. It does use a radial dependence, so it does incorporate higher-order terms in the expansion to the extent that the Taylor-expansion terms from the WRS84 expansion match that radial expansion.

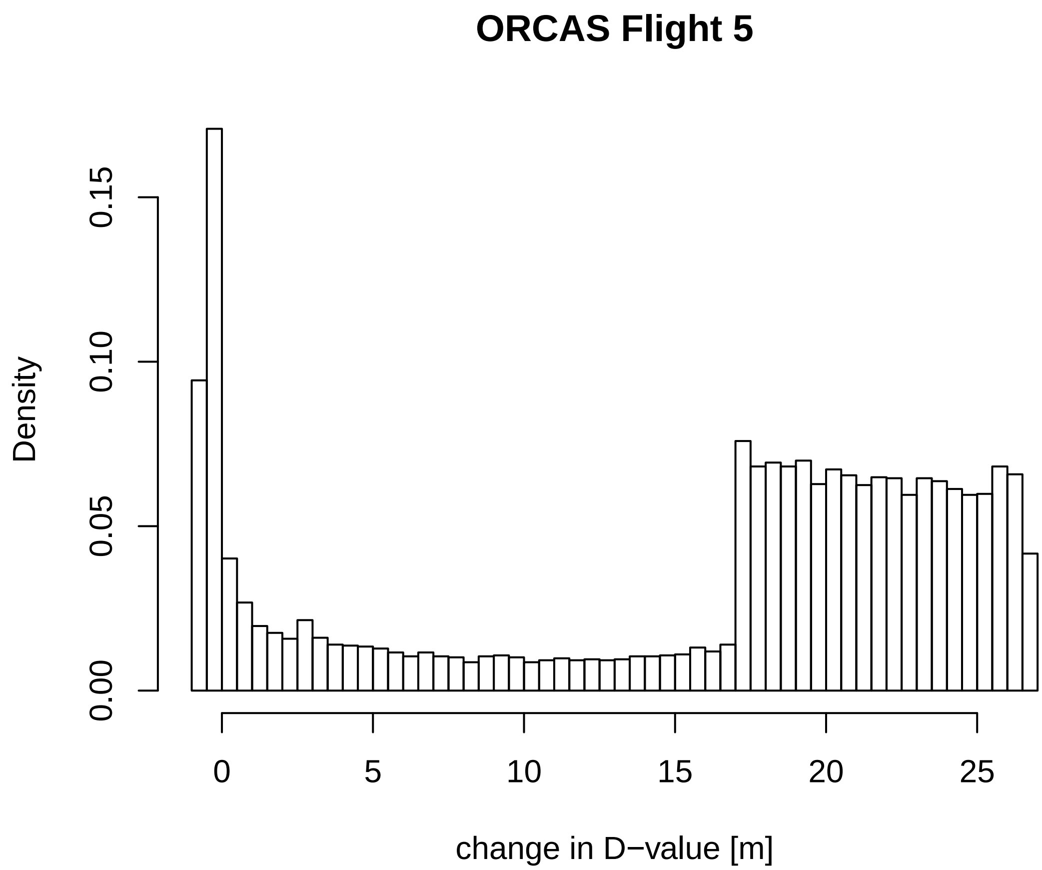
Assessment

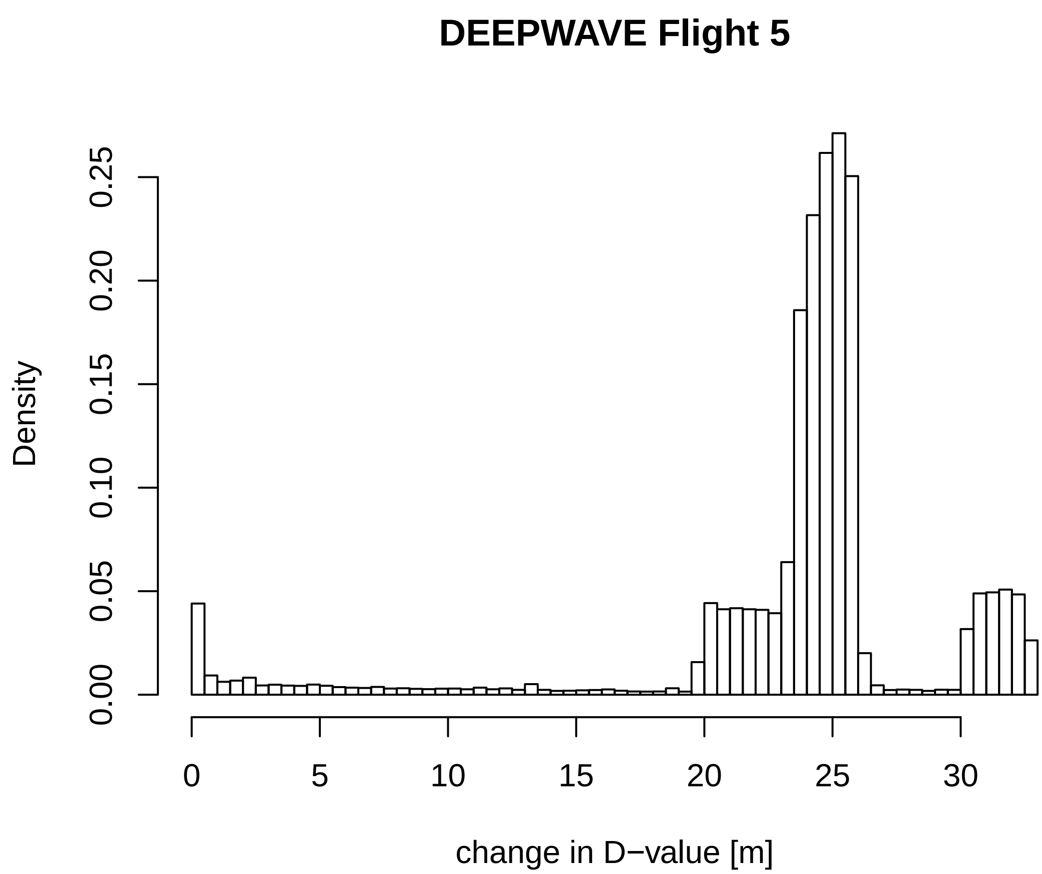
Figure 1 shows the difference between geopotential and geometric height, as a function of geometric height, for locations at several atitudes. The difference is several decameters at the upper levels and shows a strong gradient with altitude there, so the effect on measurements of d-value could

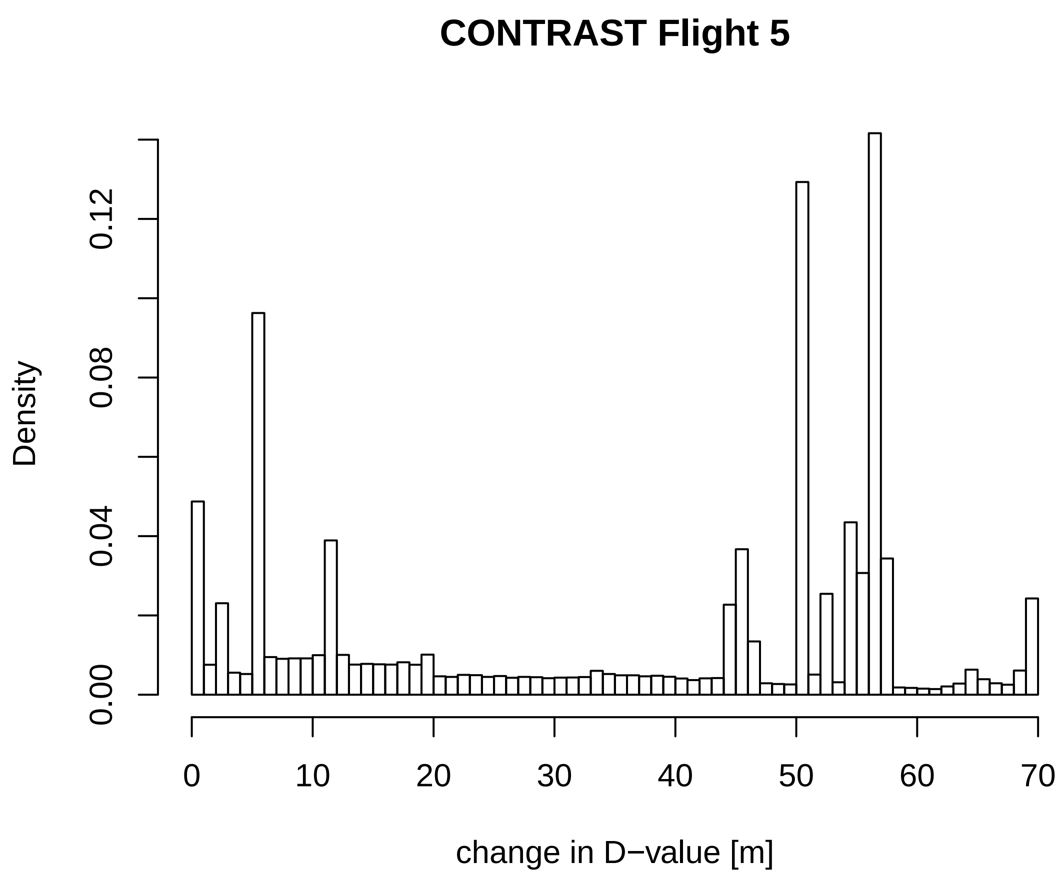


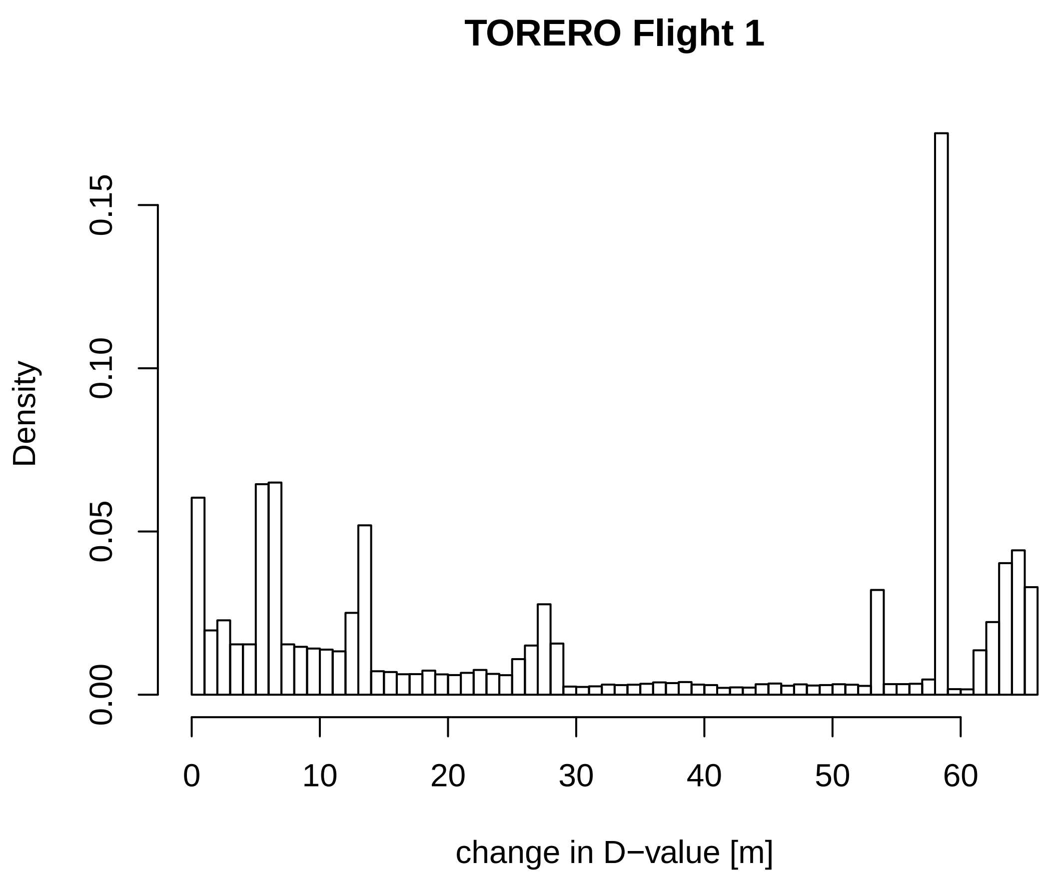
be significant both in terms of the general magnitude and with regard to gradients while flying on constant-pressure surfaces.

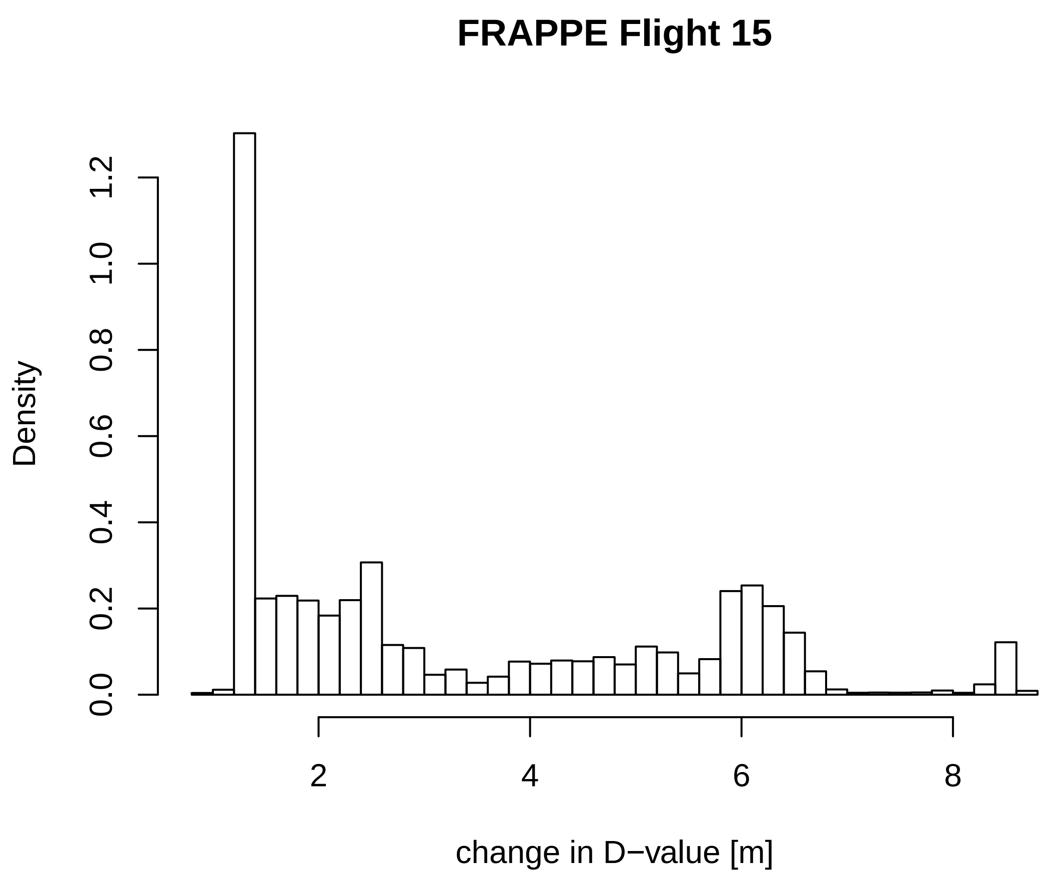
The change in d-value resulting from this change is $\Delta D = H(z,\lambda) - z$. The following histograms show that, for some representative research flights. the changes sometimes amount to several 10s of meters (see the plot for CONTRAST or TORERO) and so are significant in comparison to the expected uncertainty in the measurement of d-value.


ORCAS Flight 5
Density
0.00
0.05
0.10
0.15
0
5
10
15
20
25
change in D−value [m]


DEEPWAVE Flight 5
Density
0.00
0.05
0.10
0.15
0.20
0.25
0
5
10
15
20
25
30
change in D−value [m]


CONTRAST Flight 5
Density
0.00
0.04
0.08
0.12
0
10
20
30
40
50
60
70
change in D−value [m]


TORERO Flight 1
Density
0.00
0.05
0.10
0.15
0
10
20
30
40
50
60
change in D−value [m]


FRAPPE Flight 15
Density
0.0
0.2
0.4
0.6
0.8
1.0
1.2
2
4
6
8
change in D−value [m]

Recommendations

1. Add a new variable (perhaps GEOPTH) that represents the geopotential height. The appropriate formula is (3), where H is GGALT, λ is the latitude, Δ is GGEOIDHT, and the constants are defined above. Double precision is justified for the coefficients. Describe it as "Geopotential height [m MSL]".

2. Add a variable GGHWGS=GGALT+GGEOIDHT for convenient calculation of g from the definition in (2).

3. Change the calculation of DVALUE to be DVALUE = GEOPTH - PALT. Describe it as "D-Value, geopotential height minus pressure height".

4. Provide documentation, perhaps via a revision of the ProcessingAlgorithms technical note. This may be particularly useful now that the MJ Mahoney notes have been removed from the JPL web site. They were, I thought, the definitive description and have been used, for example, in conversions from geometric to geopotential height in WRF and other places. Another factor supporting this need is the very inadequate description of geopotential height in the AMS Glossary; see this URL, which has a two-order-of-magnitude error in the quoted value and other errors regarding significance.

—- END —-