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## Background

A quantity often called the equivalent potential temperature is used to represent stability for adiabatic wet processes and to correspond to wet adiabats on thermodynamic diagrams. Potential temperatures are generally defined to correspond to specific entropy (i.e., entropy per mass of dry air) via the relationship  $c_{px}d\ln\Theta_x = ds$  where “ $x$ ” indicates that various kinds of potential temperatures can be defined that satisfy this relationship for different processes. The AMS glossary restricts the term equivalent potential temperature to true adiabatic processes, where any condensed water remains in the parcel, but the term is often used (and has been used in Bulletin 9 Appendix B) to refer instead to the modified pseudo-adiabatic form (sometimes called pseudo-equivalent potential temperature or pseudo-adiabatic equivalent potential temperature) in which the specific heat capacity of any condensed water is neglected in the adiabatic process. This is equivalent to assuming that any condensed water falls from or is otherwise removed from the parcel as it condenses. In this memo, I will refer to the former potential temperature as the wet-equivalent potential temperature (symbol  $\Theta_q$ ) and the latter as the pseudo-adiabatic equivalent potential temperature (symbol  $\Theta_p$ ).<sup>1</sup>

The wet-equivalent potential temperature is actually easier to derive and calculate because it is related to true adiabatic processes. It is defined as that temperature that would be reached if the parcel is lifted adiabatically until all water vapor is condensed, then the dry air only is returned adiabatically to a reference pressure (conventionally 1000 hPa). For a saturated parcel, it is defined (Emanuel, 1994, p. 120, but with some changes in terminology) as:

$$\Theta_q = T \left( \frac{p_0}{p_d} \right)^{R_d/c_{pt}} \exp \left( \frac{L_v r}{c_{pt} T} \right) \quad (1)$$

where  $p_d$  is the pressure of dry air in the parcel (i.e.,  $p - e$  where  $e$  is the water-vapor pressure),  $R_d$  the gas constant for dry air,  $L_v$  the latent heat of vaporization of liquid water,  $r$  the mixing ratio of water vapor,  $T$  the (absolute) temperature, and  $c_{pt} = c_{pd} + r_t c_w$  where  $c_{pd}$  is the specific heat of dry air at constant pressure,  $c_w$  the specific heat of liquid water, and  $r_t = r + r_w$  is the total water mixing ratio and includes the contribution from the liquid water mixing ratio  $r_w$ . The assumption made is that no ice is present or forms so that all condensation is to liquid water and remains liquid water.<sup>2</sup> If the parcel is not saturated, one can lift the parcel adiabatically to its condensation point

<sup>1</sup>Pseudo-equivalent potential temperature seems to me to be a poor term because the compromise is in the degree to which the process is adiabatic, not the degree to which it is equivalent via a moist process. The term pseudo-adiabatic equivalent potential temperature, as used by e.g. Davies-Jones (2009), although lengthy, is unambiguous and so is the choice I have made. This memo uses the symbol  $\Theta_p$  instead of  $\Theta_e$  to avoid ambiguity with the definition in the AMS Glossary.

<sup>2</sup>A similar potential temperature that applies to the condensate being ice can be constructed readily by straightforward replacements, including the latent heat of sublimation, the specific heat of ice, and the vapor pressure in equilibrium with an ice surface.

and use the values of  $T$  and  $p_d$  at that point, or alternately include another term in (1) that includes a correction dependent on the relative humidity (cf. Emanuel 1994 Eq. 4.5.11).

A complication in the equations for  $\Theta_p$  or  $\Theta_q$  is that some of the material properties involved, including  $c_{pt}$ ,  $c_w$ , and  $L_v$ , are not constant but vary with temperature. An accurate integration along the adiabat, as needed to maintain the equivalence to entropy, would take into account that variation. For  $\Theta_p$ , some generalized equations have been developed that treat the variation in  $L_v$  in various ways, mostly by using an adjusted value for  $L_v$  that produces the least compromise in accuracy. See esp. Bolton (1980) and Davies-Jones (2009). Emanuel (1994) and the preceding two references present derivations of the pseudo-adiabatic equivalent potential temperature.

Because I want to perform integrations that include representations of the temperature dependence of the specific heats and latent heat as well as the Murphy and Koop (2005) representation of equilibrium water vapor pressure, I will develop the appropriate differential form of the entropy to use in such integrations. For an adiabatic process where all changes occur in equilibrium, the molar entropy  $s'$  is related to temperature  $T$  and pressure  $p$  (for a perfect gas) via

$$T ds' = c'_p dT - v' dp \quad (2)$$

where primed quantities refer to molar quantities, such that  $c'_p$  and  $v'$  are the molar heat capacity at constant pressure and the molar volume, respectively. There are three contributions to the entropy to consider for a moist air parcel: the entropy of the (i) dry air; (ii) water vapor; and (iii) liquid water. If the respective mole numbers of these three components are  $n'_d$ ,  $n'_v$ , and  $n'_w$ , then

$$T ds' = (n'_d c'_{pd} + n'_v c'_{pv} + n'_w c'_w) dT - n'_d v'_d dp_d - n'_v v'_v de + L'_v dn'_v \quad (3)$$

where  $p_d$  is the pressure of dry air,  $e$  is the water vapor pressure, and  $L'_v$  is the molar latent heat of vaporization of liquid water. The last term in (3) arises because there is an entropy change associated with the phase change from liquid water to water vapor, and this last term is the heat released by that phase change.<sup>3</sup> The other terms arise from summing (2) for the three individual components.

The mixing ratios  $r$  and  $r_w$  are, respectively, the masses of water vapor and liquid water per unit mass of dry air:

$$\begin{aligned} r &= \frac{n'_v M_w}{n'_d M_d} \\ r_w &= \frac{n'_w M_w}{n'_d M_d} \end{aligned} \quad (4)$$

where  $M_w$  is the molecular weight of water (mass of water per mole) and  $M_d$  that of dry air. Dividing (3) by  $n'_d M_d T$  and setting  $ds'$  to zero for an isentropic process leads to

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<sup>3</sup>An additional contribution arises from the entropy increase associated with mixing of the water vapor and the dry air, but this is insignificant and will be neglected.

$$(c_{pd} + rc_{pv} + r_w c_w) \frac{dT}{T} - \frac{v_d}{T} dp_d - r \frac{v_v}{T} de + \frac{L_v}{T} dr = 0 \quad (5)$$

where unprimed quantities  $c_{pd}$ ,  $c_{pv}$ ,  $c_w$ ,  $v_d$ ,  $v_v$ , and  $L_v$  are specific quantities (i.e., per unit mass of dry air for  $c_{pd}$  and  $v_d$  and per unit mass of water for  $c_{pv}$ ,  $c_w$ ,  $v_v$  and  $L_v$ ; e.g.,  $c_{pd} = c'_{pd}/M_d$  and  $c_{pv} = c'_{pv}/M_w$ ). For perfect gases,  $v_d/T = R_d/p_d$  and  $v_v/T = R_w/e$  where  $R_d$  and  $R_w$  are the gas constants for dry air and water vapor, respectively. Also, the ideal-gas form of the Clausius-Clapeyron equation is

$$\frac{de_s}{e_s} = \frac{L_v dT}{R_w T^2} \quad (6)$$

and Kirchoff's equation (cf., e.g., Emanuel 1994, Eq. 4.4,3) is

$$dL_v = (c_{pv} - c_w) dT. \quad (7)$$

With  $e = e_s(T)$  and  $r = r_s(T) = \frac{M_w}{M_d} e_s(T)/p_d$  as corresponds to a saturated parcel, and with some additional transformations as follow, all terms in (5) can be transformed into differential relationships that only involve derivatives of T and p:

$$\frac{L_v dr}{T} = \frac{d(L_v r)}{T} - r \frac{dL_v}{dT} \frac{dT}{T} = \frac{d(L_v r)}{T} - r(c_{pv} - c_w) \frac{dT}{T}$$

$$r R_w \frac{de_s}{e_s} = r R_w \frac{L_v dT}{R_w T^2} = \frac{L_v r}{T} \frac{dT}{T}$$

$$\frac{d(L_v r)}{T} = d \left( \frac{L_v r}{T} \right) + \frac{L_v r}{T} \frac{dT}{T}$$

$$d \left( \frac{L_v r}{T} \right) = \left( \frac{\partial \left( \frac{L_v r}{T} \right)}{\partial T} \right)_{p_d} dT + \left( \frac{\partial \left( \frac{L_v r}{T} \right)}{\partial p_d} \right)_T dp_d = \frac{\epsilon T d \left( \frac{L_v e_s(T)}{T} \right)}{p_d dT} \frac{dT}{T} - \frac{L_v r}{T} \frac{dp_d}{p_d}$$

where the transformation to dependence on  $T$  is a consequence of assuming that the parcel remains saturated, and where the temperature dependence of  $L_v$  and the specific heats is implicit. Gathering terms in (5) after these transformations leads to

$$\left[ (c_{pd} + r c_w) + \frac{T \epsilon}{p_d} \left( \frac{\partial \left( \frac{L_v e_s(T)}{T} \right)}{\partial T} \right)_{p_d} \right] \frac{dT}{T} = \left[ R_d + \frac{L_v r}{T} \right] \frac{dp_d}{p_d}$$

where  $r_t = r + r_w$  is the total water mixing ratio and where  $\varepsilon = M_w/M_d$ . The result then gives a relationship between  $T$  and  $p_d$ :

$$\frac{dT}{dp_d} = \frac{TR_d + L_v r}{p_d} \left[ (c_{pd} + r_t c_w) + \frac{T\varepsilon}{p_d} \left( \frac{\partial \left( \frac{L_v e_s(T)}{T} \right)}{\partial T} \right) \right]^{-1} \quad (8)$$

The result is a derivative that can be used for numerical integrations that take into account the temperature dependence of the specific heats and the latent heat of vaporization and improved representation of the equilibrium vapor pressure  $e_s(T)$ , as in Murphy and Koop (2005). This is used below to evaluate the accuracy of representations of the equivalent potential temperatures.

Equation (8) is appropriate for the adiabatic process and so can lead to the wet-equivalent potential temperature  $\Theta_q$ . The similar formula for the pseudo-adiabatic equivalent potential temperature  $\Theta_p$  can be obtained by neglecting the heat capacity of the liquid water, and so would be the same as (8) but with  $r_t$  replaced by  $r$ .

## Present Processing Code

The code from subroutine thetae.c includes the following code (where atx is converted to units of kelvin earlier in the code):

```

tlcl = (2840.0 / (3.5 * log((double)atx) - log((double)edpc) - 4.805)) + 55.0;
...
theta = atx * pow((double)1000.0 / psxc, (double)0.28571);
exparg = ((3.376 / tlcl) - 0.00254) * (mr * (1.0 + (0.00081 * mr)));
...
PutSample(varp, theta * exp((double)exparg));

```

This is based on equation (38) from Bolton (1980):

$$\Theta_E = \Theta \exp \left\{ \left( \frac{3.376}{T_L} - 0.00254 \right) r (1 + 0.81 \times 10^{-3} r) \right\} \quad (9)$$

where

$$T_L = \frac{2840}{3.5 \ln T_K - \ln e - 4.805} + 55 \quad (10)$$

is the temperature at the LCL (Bolton 1980, Eq. (21)).

## Reasons For Proposing Changes

1. The equation selected from Bolton (1980) is the less accurate of several alternatives presented there. Davies-Jones (2009) has pointed out that the form given as equation (39) in Bolton is more accurate, and Bolton also notes that.
2. Bolton (1980) also offers a different form for  $\Theta$  that is more accurate than the one now in use. His more accurate formula is:

$$\Theta_p = \Theta_{DL} \exp \left\{ \left( \frac{3.036}{T_L} - 0.00178 \right) r (1 + 0.448 \times 10^{-3} r) \right\} \quad (11)$$

where

$$\Theta_{DL} = T_k \left( \frac{1000}{p_d} \right)^{0.2854} \left( \frac{T_K}{T_L} \right)^{0.28 \times 10^{-3} r} \quad (12)$$

is the potential temperature of dry air at the lifting condensation level (LCL) and  $T_L$  is the temperature at the LCL as given by (10). Equation (11) takes into account the effect of humidity on the ratio of specific heats of the water-saturated air parcel. This form appears to be a better choice than the one now in use.<sup>4</sup>

3. The Bolton fits were based on a representation of the equilibrium water vapor pressure that is different from the one now in use in the RAF processing code. Bolton determined a fit to the data of Wexler (1976), all of which was for temperature higher than 0°C. He used extrapolation of the measurements to lower temperature, but did not test his new fit for temperatures below -35°C. Because there are now more reliable representations of the water vapor pressure (esp. Murphy and Koop 2005), it is important to check if the Bolton formula (Bolton, 1980, equation (10)) is an adequate representation of equivalent potential temperatures calculated with the new data. More significantly, the Murphy and Koop formula is based on estimates of the specific heat of liquid water in the supercooled region that vary significantly, while previous analysis like those of Bolton and of Davies-Jones (described in the next paragraph) have used constant specific heats. Some justification for use of whatever formula is used for processing is therefore needed.
4. Davies-Jones (2009) presents a new formula that appears to be slightly more accurate than the formulas of Bolton (1980), and he also optimized a large set of candidate expressions by adjusting the coefficients (including those of Bolton) to match results from exact integrations of the entropy formulas.<sup>5</sup> His new formula (Davies-Jones, 2009, Eq. (6.5)) is

$$\Theta_p^{[DJ]} = \Theta_{DL} \exp \left\{ \frac{(L_0^* - L_1^*(T_L - T_0) + K_2 r) r}{c_{pd} T_L} \right\} \quad (13)$$

<sup>4</sup>Because Bolton used fits to adjust coefficients in this formula and in (9), coefficients should be adjusted to modern values only with care. Davies-Jones (2009) has provided updated coefficients based on a similar but more extensive procedure.

<sup>5</sup>However, he has also used an approximate equation from Bolton (1980) to represent the vapor pressure.

where  $\Theta_{DL}$  is as defined by (12),  $L_0^* = 2.56313 \times 10^6 \text{ J kg}^{-1}$ ,  $L_1^* = 1754 \text{ J kg}^{-1} \text{ K}^{-1}$ , and  $K_2 = 1.137 \times 10^6 \text{ J kg}^{-1}$ . The asterisks on  $L_0^*$  and  $L_1^*$  indicate that these are the result of his optimization procedure and therefore depart slightly from the coefficients that would provide the best match to the latent heat of vaporization.

5. The name of the variable should be changed to be consistent with usage recommendations of the AMS Glossary.
6. It may be useful to include a new variable, the wet-equivalent potential temperature  $\Theta_q$ , in the data files. This is easier to calculate and less controversial than  $\Theta_p$ , and it enters analyses that have become widely used like the analyses of Paluch (1978).

Perhaps arguing against change is that, as long as we state clearly what the basis for the calculation is, users are always able to calculate a different variable of their choice from the basic measurements that are in the data set, so the variable in the data files does not constrain users. It may be preferable to retain the current equation or make small changes to represent the best values in the Bolton (1980) paper, rather than make a change, because use of the Bolton formulas has become the common convention. Changing to the higher-accuracy version from Bolton (1980) or to the Davies-Jones (2009) formula, however, would improve the accuracy of the representation of an entropy-equivalent potential temperature.

## Analysis

The desired relationship between entropy and  $\Theta_q$  is  $ds = c_{pt} d \ln \Theta_q$  where  $c_{pt} = c_{pd} + r_t c_w$ . The integral of this relationship involves an integration constant, which can be selected so that  $s$  is relative to a state with  $T = 1 \text{ K}$ ,  $r = 0$ ,  $r_w = 0$ , and  $p_d = p_0$  where  $p_0$  is the reference pressure, 1000 hPa. In the case where the specific heats are constant (but  $L_V$  still varies in accord with the Kirchhoff equation), the equation for the wet-equivalent potential temperature is

$$\Theta_q = T \left( \frac{p_0}{p_d} \right)^{R_d/c_{pt}} \exp \left\{ \frac{L_V r}{c_{pt} T} \right\} \left( \frac{e}{e_s(T)} \right)^{-r R_w/c_{pt}} \quad (14)$$

where the last term, not included in (1) but unity for a saturated parcel, arises in case of a sub-saturated parcel (below its LCL). (See the derivation in Emanuel, 1994, leading to his equation (4.5.11).)

In the case of  $\Theta_p$ , the result is not integrable even with constant specific heats because, with neglect of the heat capacity of the water carried with the parcel, the replacement for  $c_{pt}$  is  $c_{pt}^\dagger = c_{pd} + r c_w$  and this depends on  $r$ , unlike  $c_{pt}$  which is a constant. This is the reason that various approximate formulas have been developed to represent  $\Theta_p$ , including those of Bolton (1980) and Davies-Jones (2009). The desired relationship of  $\Theta_p$  to entropy is  $ds_d = C_{pd} d \ln \Theta_p$  for the dry air only.

Despite this advantage of  $\Theta_q$ , it has the disadvantage that it cannot be plotted on a thermodynamic diagram because it does not depend uniquely on  $T$  and  $p$  as do the conventional definition of potential temperature ( $\Theta$ ) and pseudo-adiabatic equivalent potential temperature (which is a unique function of  $T$  and  $p$  along saturated adiabats because the condensed water is removed so the dependence on history, which would lead to variable liquid water content in the case of  $\Theta_q$ , is lost).

The question to be addressed here is this: To what extent do relationships based on various equations for  $\Theta_p$  and  $\Theta_q$  depart from the representation of the entropy that arises from direct integration of (3) relative to the reference state? This can be addressed by evaluating the extent to which constant values of  $\Theta_p$  or  $\Theta_q$  represent true pseudoadiabats or wet adiabats during vertical motion of air parcels. Davies-Jones (2009) has addressed this question for the Bolton formulas, a new alternative he proposed, and some others. However, two considerations not discussed there are relevant to RAF applications:

- What is the effect of introduction of the Murphy and Koop (2005) representation of equilibrium water vapor pressure? This will result in small shifts in results but should not affect the analysis of accuracy as presented by Davies-Jones (2009) because the formulas have vapor pressure as an input. That analysis considers adjustment of coefficients in the formulas to compensate for factors like the variation in latent heat with temperature, and those coefficients have been minimized by comparison to numerical integrations that use old representations of vapor pressure, but this does not seem likely to have much effect on the results because the vapor-pressure formula used is close to the revised form presented by Murphy and Koop (2005) and would enter both the parametrized formula and the exact integration.
- What is the effect of temperature dependence of the specific heats, esp. that of supercooled water? The temperature dependence of the specific heats is neglected, both in the parametrized formulas and in the exact integration as done by Davies-Jones, Bolton, and others. However, the vapor-pressure formulas offered by Murphy and Koop (2005) are based on a representation of the specific heat of supercooled water that has significant variation, of order 50% overall. This applies to the wet-equivalent potential temperature as well as the pseudo-adiabatic equivalent potential temperature.

One approach to both questions is to integrate  $dT = (dT/dp_d) dp_d$  using the exact relationship (8) and compare the result to that obtained from the formulas that were obtained with constant specific heats. For example, consider a case where the temperature and dry-air pressure are  $T_1$  and  $p_{d,1}$  and the parcel is saturated. One can obtain the temperature at a low pressure  $p_{d,2}$  (e.g., 100 hPa) by evaluating the appropriate formula for  $\Theta_q$  or  $\Theta_p$ , then inverting that formula to obtain the temperature at  $p_{d,2}$  that gives the same corresponding equivalent potential temperature. Alternately, one can integrate (8) to obtain the same temperature. If the temperature dependence of the specific heats is included in the latter integration, the difference between results will provide a measure of the error in equivalent potential temperature obtained from the formula used for its evaluation.<sup>6</sup>

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<sup>6</sup>It will be useful to evaluate the formula used for the LCL also, because this will also be affected by the variation in specific heats and by the change in representation of the equilibrium vapor pressure; this has not been included here yet.

To do this, the temperature dependence of the specific heats must be represented. The data in the Appendix to this memo specify the data available. The most important variation is that of the specific heat of liquid water (including in the supercooled region), which varies quite significantly (more than 50%) in the supercooled region. Because the water vapor pressure becomes quite low for substantial supercooling, it is not clear that even this very large variation has a significant effect, but this is likely to be the dominant term causing a departure from constant- $c_p$  formulas. The specific heat of dry air, in contrast, varies little (less than 0.1% from  $-50^\circ\text{C}$  to  $+40^\circ\text{C}$ , and less than 0.4% from  $-100$  to  $+60^\circ\text{C}$ ), so that variation will be neglected here. The specific heat of water vapor does not enter the formulas for pseudo-adiabatic or wet equivalent potential temperature, but it does enter indirectly through the Kirchhoff equation (7), which relates the temperature dependence of  $L_v$  to the difference between the specific heats of water vapor and liquid water. There are significant discrepancies among values listed in the Appendix, but for this study I have used the table in Wagner and Pruss with quadratic extrapolation into the region below 0:

$$c_{pv} = 1.932468 - 1.343977 \times 10^{-3}T + 4.275412 \times 10^{-6}T^2 \quad (15)$$

For example, consider an LCL at  $T_1 = (10 + 273.15)\text{K}$  and  $p_{d,1} = 750$  hPa. Formula (1) gives  $\Theta_q = 335.47\text{K}$ , and formula (13) gives  $\Theta_p = 338.93\text{K}$ .<sup>7</sup> At  $p_{d,2} = 100$  hPa, inversion of these formulas give the respective temperatures 178.69 K and 175.66 K. For comparison, integrations from point 1 to point 2 using (8) or the analog with  $c_{pt}^\dagger = c_{pd} + rc_w$  give the respective values 178.71 K and 175.69 K, both quite close to the values obtained by the respective equations.<sup>8</sup>

The following table gives these and a few more values, where  $T_q$  and  $T_p$  are the temperatures determined at the final pressure  $p_2$  from inversion of the formulas for  $\Theta_q$  and  $\Theta_p$ , and the corresponding primed quantities are the temperatures determined by numerical integration that incorporates the variation in specific heat.:

T [ $^\circ\text{C}$ ]	p [hPa]	final $p_2$ [hPa]	$T_q$ [K]	$T'_q$ [K]	$T_p$ [K]	$T'_p$ [K]
25	850	100	206.50	207.69	200.73	200.76
15	750	100	189.34	189.73	185.39	185.42
10	750	100	178.69	178.71	175.66	175.69
0	700	100	167.05	166.90	165.39	165.41
-10	600	100	163.54	163.44	162.68	162.70
25	850	300	222.59	222.69	222.27	222.29
10	750	300	239.43	239.46	238.52	238.51
0	700	300	226.46	226.59	225.82	225.84

Although the evaluation is not exhaustive, it appears to support these conclusions:

<sup>7</sup>These values are obtained with the Davies-Jones (2009) values of  $c_{pd} = 1005.7 \text{ J kg}^{-1} \text{ K}^{-1}$ ,  $c_{pv} = 1875 \text{ J kg}^{-1} \text{ K}^{-1}$ , and  $L_v = L_0 + L_1(T - T_0)$  where  $L_0 = 2.501 \times 10^6 \text{ J kg}^{-1}$  and  $L_1 = 2370 \text{ J kg}^{-1} \text{ K}^{-1}$ .

<sup>8</sup>The integrations with constant specific heat consistently produced agreement with the formulas to an accuracy of 0.01 K. This provides a check on the accuracy of the integration, but also shows consistency with the optimized coefficients of Davies-Jones (2009), because the integration is independent of the coefficients he determined.



1. Comparison of the last two columns suggests that the formula (13) represents the pseudo-adiabats with good accuracy, even when they are calculated using the strong variations in the specific heat of liquid water estimated by Murphy and Koop (2005) and using the Murphy and Koop vapor pressures.
2. Even for the reversible adiabatic equation (1), there is only minor deviation from true adiabats when the temperature variation of the specific heats is represented accurately. The variations from the temperature on a true wet adiabat are only a few tenths degree in the worst cases. One might expect more variation, because high liquid water content can be carried upward where the variation in specific heat of liquid water can affect the answer. These are unrealistic situations, however, because high liquid water content cannot exist at temperatures as low as the extremes of this integration.

One additional issue is the representation in cases where the parcel is unsaturated. Equation (13) represents the pseudo-adiabatic equivalent potential temperature in terms that involve the saturation temperature, the temperature at the LCL. To determine this, one can use (10) from Bolton (1980), include the additional term in the entropy equation as derived by Emanuel (1994),<sup>9</sup> or find the LCL from conservation of the mixing ratio and of the moist potential temperature during adiabatic expansion. If the specific heats are functions of temperature, it is not clear what choice to make.

Emanuel (1994) also gives an equation for the saturation temperature, his Eq. (4.6.21):

$$-\ln \frac{e}{e_s(T)} = \left( \frac{c_{pd}}{R_d} \frac{(1 + rc_{pv}/c_{pd})}{(1 + r/\epsilon)} + \frac{c_w - c_{pv}}{R_w} \right) \ln \frac{T_L}{T} + \left[ \frac{L_V(T_0)}{R_w} \right] \left( \frac{1}{T_L} - \frac{1}{T} \right)$$

(with  $T_0 = 273.15$  K), which can be solved numerically for  $T_L$ . Alternately, one can determine the temperature of the rising parcel from the conserved values of mixing ratio and moist potential temperature, iteratively evaluated for the temperature that gives the equilibrium vapor pressure.

To evaluate how well the Bolton formula for  $T_L$  represents exact values, a set of calculations were performed where the LCL was determined by lifting the parcel adiabatically and using the conservation of moist potential temperature and mixing ratio during adiabatic ascent to determine the LCL numerically. The approach was as follows:

1. Determine the appropriate values of the molecular weight and specific heat of moist air using these equations:

$$M^* = \frac{1 + r}{1 + r/\epsilon} M_d$$

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<sup>9</sup>i.e., multiply the formula for pseudo-adiabatic equivalent potential temperature by

$$\left( \frac{e}{e_s(T)} \right)^{-rR_w/c'_{pt}}$$

$$c_p^* = c_{pd} \left( \frac{1}{1+r} \right) + c_{pv} \frac{r}{1+r} = c_{pd} \frac{1 + rc_{pv}/c_{pd}}{1+r}$$

$$\frac{R^*}{c_p^*} = \frac{R_d}{c_{pd}} \left( \frac{1 + r/\epsilon}{1 + rc_{pv}/c_{pd}} \right) \quad (16)$$

2. Find the potential temperature for the moist mixture from

$$\Theta_m = T \left( \frac{1000}{p} \right)^{R^*/c_p^*} \quad (17)$$

3. Use conservation of  $\Theta_m$  and the mixing ratio  $r$  to find the pressure at which the temperature deduced from (XXX) gives a saturation vapor pressure equal to the vapor pressure deduced from  $e = rp/(r + \epsilon)$ , the latter as required for a constant mixing ratio. Specifically, this was done by minimizing the following function of pressure:

$$f(p) = e_s(T(p)) - \frac{rp}{r + \epsilon}$$

where  $T(p)$  is the temperature determined from (XXX) above.

A short list of checks, shown in the following table, suggested that the Bolton formula for  $T_L$  performs well, giving values within 0.05 K of those determined by the numerical solution. Because this is a published equation in common use, there does not appear to be any need for a change.

Temperature [°C]	Pressure [hPa]	RH [%]	(10) – Bolton	Numerical
20	850	50	280.074	280.067
25	900	30	275.42	275.43
25	900	20	290.94	290.92
10	700	80	279.14	279.14
10	700	50	271.12	271.24
10	700	10	247.54	247.58
0	600	80	269.45	269.48
0	600	40	258.79	258.84
0	600	20	249.13	249.18

### ***Recommendations:***

1. Change to (13), and change the variable name to “pseudo-adiabatic equivalent potential temperature”. Use (10) to determine the saturation temperature  $T_L$ . (There will need to be some protection against the relative humidity exceeding 100%; if it does, set it to 100% for this calculation.)

2. Add a new variable “wet equivalent potential temperature” and use (14) for its evaluation. A caution should be included to warn users that there may be inaccuracies in  $\Theta_q$  of a few tenths kelvin that arise from variation and uncertainty in the specific heat of supercooled water.
3. It might be useful to consider adding still another potential temperature that refers to ice-phase processes, because that is more commonly the case at GV flight altitude. However, that is a project for another time.<sup>10</sup>

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<sup>10</sup>An approach similar to that of Davies-Jones (2009) would be appropriate, where coefficients in an equation of the form he uses are adjusted to match exact-integration values over a broad range of conditions.

## Appendix: Values Used For Specific Heats

For dry air, the following values come from the “Engineering Toolbox” and are reasonable in comparison to sparsely placed values in the CRC Handbook. They suggest that there is no significant variation in the specific heat of dry air over the range of temperatures encountered by RAF research aircraft.

T (°C)	$c_p$ [kJ kg <sup>-1</sup> K <sup>-1</sup> ]
-150	1.026
-100	1.009
-50	1.005
0	1.005
20	1.005
40	1.005
60	1.009

For water vapor, the “Engineering Toolbox” provides the following data (in conflict with the data from Wagner and Pruss (2002) presented later, and also with the Penn State summary of physical constants which gives a single value, 1.952):

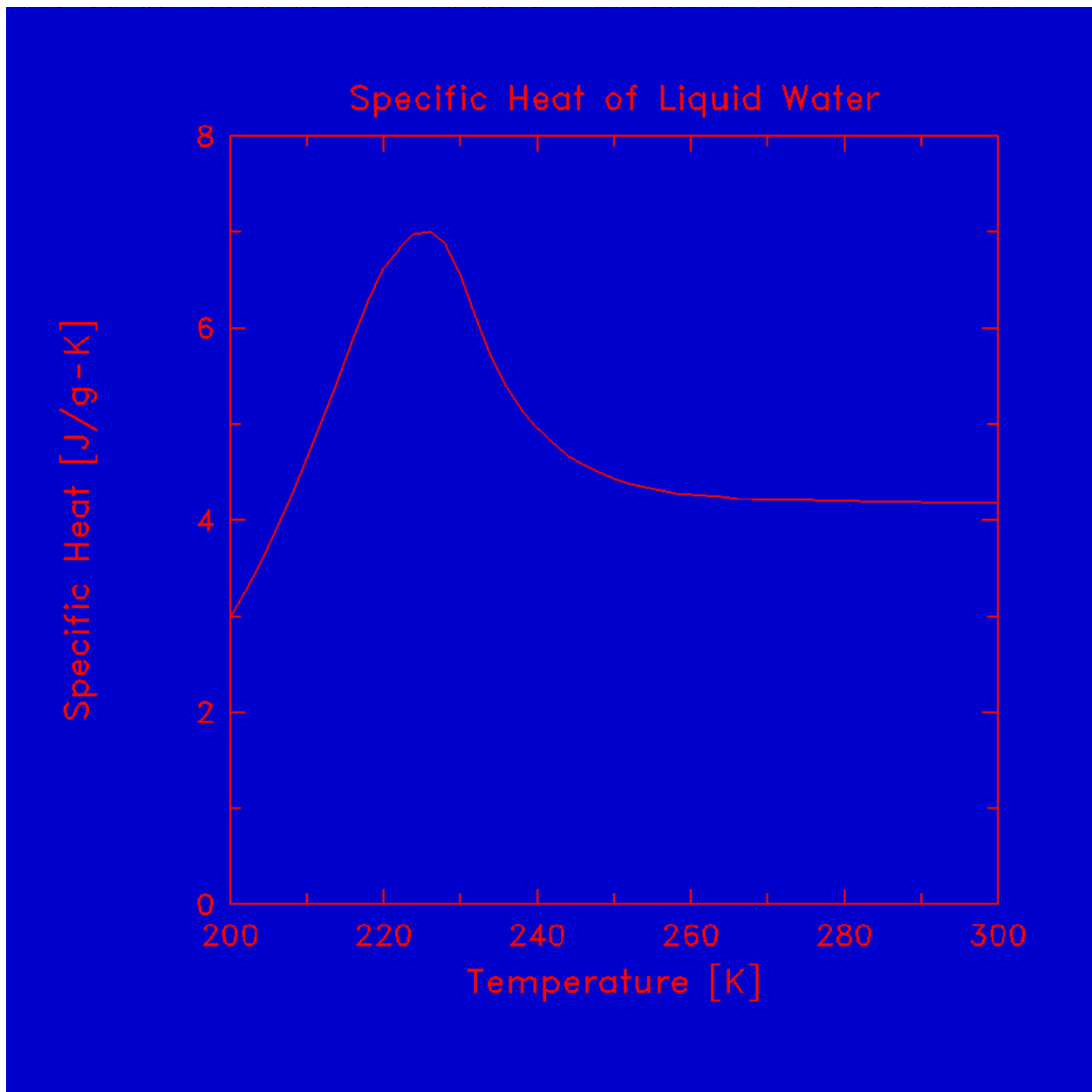
T [K]	$c_p$ [kJ kg <sup>-1</sup> K <sup>-1</sup> ]
175	1.850
200	1.851
225	1.852
250	1.855
275	1.859
300	1.871
325	1.880

For liquid water at 1000 hPa ( $M_w = 18.01528$ ),

T (°C)	$c_p$ [kJ kg <sup>-1</sup> K <sup>-1</sup> ]
0	4.2176
10	4.1921
20	4.1818
30	4.1784
40	4.1785
50	4.1806
60	4.1843

For supercooled water, Murphy and Koop (2005) presented their estimate for the specific heat in their Fig. 6, and they referenced Wagner and Pruss (2002) for the specific heat of water at temperature greater than 0°C. Tables were constructed from these two sources, giving the values below

(which will here be used in preference to those immediately above, from the CRC Handbook) as well as reference values for the specific heat of water vapor. For liquid water, the following plot represents the tabulated data. This figure and the following table show that there is significant variation in the specific heat of liquid water, especially in the supercooled-water region, and this is likely to affect the calculation of equivalent potential temperatures (whether carried with the parcel or not, because of the link between the specific heat of liquid water and the variation in the latent heat of vaporization via the Kirchoff equation).



T [K]	$c_w [kJ kg^{-1} K^{-1}]$	$c_{pv} [kJ kg^{-1} K^{-1}]$
200	2.9873	
202	3.2597	
204	3.5615	
206	3.8970	
208	4.2637	
210	4.6348	
212	5.0286	
214	5.4508	
216	5.8732	
218	6.2855	
220	6.6228	
222	6.8251	
224	6.9813	
226	7.0099	
228	6.8818	
230	6.5527	
232	6.1004	
234	5.7058	
236	5.3838	
238	5.1507	
240	4.9515	
242	4.7998	
244	4.6721	
246	4.5790	
248	4.5002	
250	4.4397	
252	4.3831	
254	4.3457	
256	4.3113	
258	4.2808	
260	4.2708	
262	4.2538	
264	4.2452	
266	4.2210	
268	4.2191	
270	4.2104	
272	4.2087	

T [K]	$c_w [kJ kg^{-1} K^{-1}]$	$c_{pv} [kJ kg^{-1} K^{-1}]$
274	4.2171	1.8852
276	4.2110	1.8872
278	4.2058	1.8893
280	4.2014	1.8913
282	4.1975	1.8935
284	4.1942	1.8956
286	4.1914	1.8978
288	4.1890	1.9000
290	4.1869	1.9023
292	4.1852	1.9046
294	4.1838	1.9069
296	4.1826	1.9092
298	4.1817	1.9116
300	4.1809	1.9141

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